On the Numerators of Hurwitz Numbers

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Definition

Hurwitz numbers represent a sequence of positive rational numbers that can be defined by the following equation:

$$H_k = \frac{(k)!}{(2\omega)^k} S_k \quad \text{where}$$

- $S_k = \sum_{x \in \mathbb{Z}[i] \setminus \{0\}} x^{-k}$
- $\omega = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}} \approx 2.622$

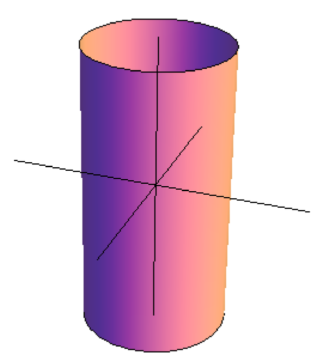
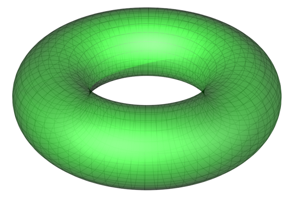
Note that $S_k = H_k = 0$ unless k is a multiple of 4.

#Bernoulli numbers

Compare Hurwitz with Bernoulli numbers, which are often described by the following:

- $\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}$
- $2\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n)!} B_{2n}$
- $\pi = 2\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Table 1: Bernoulli vs Hurwitz

B_{2n}	H_{4n}
$denom(B_{2n}) = \prod_{(p-1)\mid 2n} p$	$denom(H_{4n}) = 2 \prod_{(p-1)\mid 4n} p$
Fractional decomposition (Von Staudt-Clausen Thm): $(B_{2n} + \sum_{(p-1)\mid 2n} \frac{1}{p}) \in \mathbb{Z}$	Fractional decomposition (Hurwitz): $H_{4n} = G_{4n} + \frac{1}{2} + \sum \frac{(2a_p)^{\frac{4n}{p-1}}}{p}$ where $G_{4n} \in \mathbb{Z}$ and $p = (a_p + b_p i)(a_p - b_p i)$
Taylor expansion of a trigonometric function $tan(x) = \frac{sin(x)}{cos(x)} = \sum(-1)^{n-1} 2^{2n}(2^{2n}-1) B_{2n} \frac{x^{2n-1}}{(2n)!}$	Taylor expansion of the Weierstrass $\wp$-function (over the lattice $L = \omega \cdot \mathbb{Z}[i]$): $\wp(z) - \frac{1}{z^2} = \sum_{\lambda \in L \setminus \{0\}} (\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}) = \sum \frac{2^{4n} H_{4n}}{4n} \cdot \frac{z^{4n-2}}{(4n-2)!}$
$\phi_1 : z \mapsto (sin(z), cos(z))$  **Cylinder**: periodic	$\phi_2 : z \mapsto (\wp(z), \wp'(z))$  **Torus**: doubly-periodic

First 15 H_{4n}'s

Each numerator has a "tame part" consisting of many small primes $\equiv 3 \pmod 4$ and a "wild part" (**bolded**) of sporadic primes $\equiv 1$ or $3 \pmod 4$.

k	H_k
4	$\frac{1}{10}$
8	$\frac{3}{10}$
12	$\frac{3^4 \cdot 7}{130}$
16	$\frac{3^4 7^2 \cdot 11}{170}$
20	$\frac{3^6 7^2 \cdot 11}{10}$
24	$\frac{3^7 7^3 11^2 \cdot 19}{130}$
28	$\frac{3^9 7^4 11^2 \cdot 19 \cdot 23}{290}$
32	$\frac{3^{10} 7^4 11^2 \cdot 19 \cdot 23 \cdot \mathbf{223}}{170}$
36	$\frac{3^{14} 7^5 11^3 \cdot 19 \cdot 23 \cdot 31 \cdot \mathbf{61}}{4810}$
40	$\frac{3^{13} 7^5 11^3 19^2 \cdot 23 \cdot 31 \cdot \mathbf{2381}}{410}$
44	$\frac{3^{15} 7^6 11^4 19^4 \cdot 23 \cdot 31}{10}$
48	$\frac{3^{16} 7^5 11^4 19^2 23^2 \cdot 31 \cdot 43 \cdot \mathbf{1162253}}{2210}$
52	$\frac{3^{18} 7^7 11^4 \cdot \mathbf{13} \cdot 19^2 23^2 \cdot 31 \cdot 43 \cdot 47 \cdot \mathbf{8887}}{530}$
56	$\frac{3^{19} 7^8 11^5 19^2 23^2 \cdot 31 \cdot 43 \cdot 47 \cdot \mathbf{61} \cdot \mathbf{52289}}{290}$
60	$\frac{3^{22} 7^8 11^5 19^3 23^2 \cdot 31 \cdot 43 \cdot 47 \cdot \mathbf{2630966033}}{7930}$

Table 2: First 15 H_{4n}'s

Remark

If
- $p \equiv 3 \pmod 4$
- $p < k - 4$,

then $p \mid num(H_k)$

Define

- p-adic valuation: $\boldsymbol{v_p(n)} = max\{v \in \mathbb{N} : p^v \mid n\}$
- $\boldsymbol{f_p(n)} = v_p(num(H_{4n}))$, e.g. $f_3(15) = 22$
- $\boldsymbol{d_p(n)} = f_p(n+1) - f_p(n)$

Results for $p \equiv 1 \pmod 4$

Denote $p' = \frac{p-1}{4}$. If $p' \mid n$, then $p \mid denom(H_{4n})$, so $f_p(n) = 0$. We assume $p' \nmid n$.

The Only Two Categories of p

I. p is regular if $f_p(n) = 0 \iff gcd(n,p) = 1$

II. p is irregular if $\exists\, n_0$ where $0 < n_0 < p'$ such that $f_p(n_0) > 0$

p	61	109	197	277	337
n_0	9, 14	16, 25	47	63	26, 28, 71

Table 3: First 5 irregular primes with their n_0's

Congruence (Katz)

$$\frac{H_{k+p-1}}{k+p-1} \equiv 2a_p \frac{H_k}{k} \pmod p$$

where $2a_p \equiv \frac{3 \cdot 7 \cdot 11 \cdots (p-2)}{1 \cdot 5 \cdot 9 \cdots (p-4)} \not\equiv 0 \pmod p$

$$\therefore f_p(n) \begin{cases} = v_p(n) & \text{if } p: \text{regular} \\ = v_p(n) & \text{if } n \not\equiv n_0 \pmod{p'} \text{ for } p: \text{irregular} \\ > v_p(n) & \text{if } n \equiv n_0 \pmod{p'} \text{ for } p: \text{irregular} \end{cases}$$

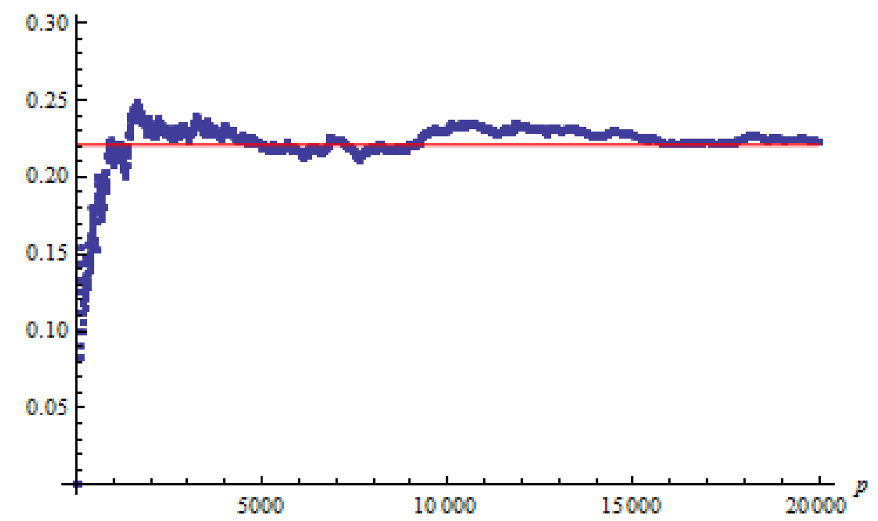
Additional Results

Growth of H_{4n}

$$\frac{H_{4n}}{H_{4n-4}} \sim \frac{(4n)(4n-1)(4n-2)(4n-3)}{(2\omega)^4}$$

Compare with the growth of B_{2n}: $\frac{B_{2n}}{B_{2n-2}} \sim \frac{(2n)(2n-1)}{(2\pi)^2}$

Density of Irregular Primes



Conjectured density: $1 - e^{-1/4} \approx 22.12\%$

Algorithms

Recursive Formula

From $(\wp')^2(u) = 4\wp^3(u) - 4\wp(u)$, we have the recursion:

Let $J_1 = \frac{1}{80}$ and $J_n = \frac{6}{(4n+1)(4n-6)} \sum_{0<m<n} J_m J_{n-m}$

$$\implies H_{4n} = \frac{(4n)!}{4n-1} J_n$$

Direct Formula

Approximate S_{4n} with enough precision where

$$S_{4n} = \sum_{x \in \mathbb{Z}[i] \setminus \{0\}} x^{-4n} \approx \prod_{\substack{p = \text{Gaussian prime} \\ Re(p) > 0,\ Im(p) \geqslant 0}} \frac{4}{1 - p^{-4n}}$$

Estimate G_{4n}, the integer part of H_{4n}, by

$$G_{4n} = H_{4n} - \frac{1}{2} - \sum \frac{(2a_p)^{\frac{4n}{p-1}}}{p}$$

Add the integer part and the fractional part, which is known, to compute the exact value of H_{4n}.

Results for $p \equiv 3 \pmod 4$

Case Study: $p = 3$

$d_3(n)$	+0	+6	+12	+18	+24	+30
1	1	1	1	1	1	1
2	3	4	3	3	5	3
3	0	-1	0	0	-2	0
4	2	2	2	2	2	2
5	1	1	1	1	1	1
6	2	2	2	2	2	2

Table 4: Interesting pattern for $d_3(n)$ (*reads vertically*)

$\Rightarrow$

3	3	3	3	3	3
4	5	4	4	6	4
3	3	3	3	3	3

$\Rightarrow$

4	4	4	4	4	4
5	6	5	5	7	5
4	4	4	4	4	4

$\therefore$ Conjecture (*verified up to* $4n = 60000$):

$$f_3(6j + a) = \begin{cases} 9j + f_3(a) + v_3(6j+a) - v_3(a) & (0 < a < 6) \\ 9j + f_3(a) & (a = 6) \end{cases}$$

Conjecture for General $f_p(n)$

$$f_p(t_p j + a) = \begin{cases} p^2 j + f_p(a) + v_p(t_p j + a) - v_p(a) & (0 < a < t_p) \\ p^2 j + f_p(a) & (a = t_p) \end{cases}$$

where t_p = "pseudo-period" = $\frac{p(p^2-1)}{4}$

- Ex: $f_7(90) = f_7(84 \cdot 1 + 6) = 7^2 \cdot 1 + f_7(6) + v_7(90) - v_7(6) = 49 + 3 + 0 - 0 = 52$

Growth of $f_p(n)$

$$\frac{f_p(n)}{n} \sim \frac{4p}{p^2 - 1}$$

Mystery Numbers

The conjecture is true except for some n's where the formula deviates by 1 or 2 for some fixed p. In fact, these mystery numbers are related to the zeroes of a p-adically interpolated L-function.

Table 5: Mystery numbers

$p = 3$	7	11	19	23	31
X	573	2125, 3584	4183, 11241	7126	11621

References

[1] John L. Boxall. A new construction of p-adic L-functions attached to certain elliptic curves with complex multiplication. *Annales de l'institut Fourier*, 36(4):31–68, 1986.

[2] A. Hurwitz. Ueber die entwickelungscoefficienten der lemniscatischen functionen. *Mathematische Annalen*, 51(2):196–226, 1898.

[3] Nicholas M. Katz. The congruences of Clausen - von Staudt and Kummer for Bernoulli-Hurwitz numbers. *Mathematische Annalen*, 216(1):1–4, 1975.